Characterizations and Dispersion-Matrix Robustness of Efficiently Estimable Parametric Functionals in Linear Models with Nuisance Parameters

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ABSTRACT

Algebraic representations, dimensional expressions, and characterizations are given for the subspace of functionals of the main parameters γ, which can be estimated with full efficiency under a linear model $(\mathbf{y}, \mathbf{W}\gamma + \mathbf{Z}\delta, \sigma^2\mathbf{V})$ containing nuisance parameters δ. Subspaces of functionals of γ, for which the ordinary least-squares estimator is robust against an alternative dispersion matrix $\mathbf{V}$, are obtained, and a particular subspace of such functionals is found wherein the ordinary least-squares estimator is both dispersion-matrix robust and robust against the presence of nuisance parameters.

1. INTRODUCTION

Let $\mathbb{R}^p$ and $\mathbb{R}^{p\times q}$ denote the set of p-dimensional real column vectors and the set of $p \times q$ real matrices, respectively. In addition, let $\dim \mathscr{U}$, $\mathscr{V} \oplus \mathscr{W}$, and $\mathscr{V} \boxplus \mathscr{W}$ stand for the dimension of a subspace $\mathscr{U}$, the direct sum of two subspaces $\mathscr{V}$ and $\mathscr{W}$, and the orthogonal direct sum of $\mathscr{V}$ and

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$\mathscr{W}$, respectively. Given $\mathbf{A} \in \mathbb{R}^{p \times q}$, the symbols $\mathbf{A}'$, $\mathscr{R}(\mathbf{A})$, $\mathscr{N}(\mathbf{A})$, and $\mathrm{r}(\mathbf{A})$ will denote the transpose, range, null space, and rank, respectively, of $\mathbf{A}$. Further, $\mathbf{A}^-$ will denote an arbitrary generalized inverse of $\mathbf{A}$ satisfying $\mathbf{A}\mathbf{A}^-\mathbf{A} = \mathbf{A}$, and $\mathbf{A}^\perp$ will stand for any matrix such that $\mathscr{R}(\mathbf{A}^\perp) = \mathscr{R}^\perp(\mathbf{A})$, the orthocomplement of $\mathscr{R}(\mathbf{A})$. Moreover, $\mathbf{P}_\mathbf{A}$ and $\mathbf{Q}_\mathbf{A}$ will denote the orthogonal projectors onto $\mathscr{R}(\mathbf{A})$ and $\mathscr{R}^\perp(\mathbf{A})$, respectively.

For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{p \times p}$, we shall write $\mathbf{A} \geqslant_\mathrm{L} \mathbf{B}$ whenever $\mathbf{A} - \mathbf{B}$ is symmetric and nonnegative definite (n.n.d.). This is the usual Löwner partial ordering of $\mathbb{R}^{p \times p}$ [cf. Löwner (1934, p. 177) and also Marshall and Olkin (1979, p. 462)]. For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{p \times q}$, we shall also write $\mathbf{A} \geqslant_{\mathrm{rs}} \mathbf{B}$ whenever $\mathrm{r}(\mathbf{A} - \mathbf{B}) = \mathrm{r}(\mathbf{A}) - \mathrm{r}(\mathbf{B})$. As shown by Hartwig (1980, Theorem 2), this relation defines a partial ordering of $\mathbb{R}^{p \times q}$, usually referred to as the rank-subtractivity partial ordering.

Consider the linear models $\mathscr{M}_a(\mathbf{I}) = (\mathrm{y}, \mathbf{W}\boldsymbol{\gamma} + \mathbf{Z}\boldsymbol{\delta}, \sigma^2\mathbf{I})$ and $\mathscr{M}(\mathbf{I}) = (\mathrm{y}, \mathbf{W}\boldsymbol{\gamma}, \sigma^2\mathbf{I})$, in which y is an observable random vector with expectations $E_a(\mathrm{y}) = \mathbf{W}\boldsymbol{\gamma} + \mathbf{Z}\boldsymbol{\delta}$ and $E(\mathrm{y}) = \mathbf{W}\boldsymbol{\gamma}$, respectively; $\mathbf{W} \in \mathbb{R}^{n \times m}$ and $\mathbf{Z} \in \mathbb{R}^{n \times k}$ are known (nonzero) model matrices of arbitrary rank; $\boldsymbol{\gamma} \in \mathbb{R}^m$ and $\boldsymbol{\delta} \in \mathbb{R}^k$ are unknown vectors of main parameters and nuisance parameters, respectively; and $D(\mathrm{y}) = \sigma^2\mathbf{I}$ is the dispersion (variance-covariance) matrix of y. Linear models of the type $\mathscr{M}_a(\mathbf{I})$, where only a part of the vector of location parameters is of interest to the experimenter, are encountered in many areas of application. Well-known examples of such models are the various models related to experimental designs such as block or row-column designs, where $\boldsymbol{\gamma}$ corresponds to the treatment effects and $\boldsymbol{\delta}$ comprises the block or row and column effects.

The earliest systematic investigation of a partitioned linear model appears to have been by Rao (1946). Another precursor is Ehrenfeld (1955a), who studied the effect of the presence of nuisance parameters on the precision of the best linear unbiased estimator (BLUE) of an arbitrary functional $\mathbf{p}'\boldsymbol{\gamma}$. Ehrenfeld's main result was later extended by Fellman (1976), who dropped the restrictive assumption of a full-rank model matrix $(\mathbf{W} : \mathbf{Z})$. He also gave an explicit characterization of the set $\mathscr{E}_a$, comprising all the functionals $\mathbf{p}'\boldsymbol{\gamma}$ which are (linearly and unbiasedly) estimable under the model $\mathscr{M}_a(\mathbf{I})$.

An exhaustive comparison between the models $\mathscr{M}_a(\mathbf{I})$ and $\mathscr{M}(\mathbf{I})$ has been made by Baksalary (1984). Additional results and comments, with special reference to the commutativity of certain orthogonal projectors, may be found in Baksalary (1987). Another recent paper by Fellman (1985) exhibits a characterization of the subspace $\mathscr{E}_0(\mathbf{I}) \subset \mathscr{E}_a$, comprising all those $\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a$ which can be estimated with the same precision under the models $\mathscr{M}_a(\mathbf{I})$ and $\mathscr{M}(\mathbf{I})$. The only paper considering a possibly singular dispersion matrix, in the above context, appears to be Kubáček (1986), where the results are, however, derived throughout under the restrictive disjointness assumption $\mathscr{R}(\mathbf{W}) \cap \mathscr{R}(\mathbf{Z}) = \{\mathbf{0}\}$.

In this paper we compare the linear models $\mathscr{M}_a(\mathbf{V}) = (\mathbf{y}, \mathbf{W}\boldsymbol{\gamma} + \mathbf{Z}\boldsymbol{\delta}, \sigma^2\mathbf{V})$ and $\mathscr{M}(\mathbf{V}) = (\mathbf{y}, \mathbf{W}\boldsymbol{\gamma}, \sigma^2\mathbf{V})$, allowing $\mathbf{V}$ to be singular such that

$$\mathscr{R}(\mathbf{W}:\mathbf{Z}) \subset \mathscr{R}(\mathbf{V}), \tag{1.1}$$

thus extending the abovementioned results obtained for the special case $\mathbf{V} = \mathbf{I}$. Note that, under the assumption (1.1), $\mathbf{y} \in \mathscr{R}(\mathbf{V})$ (a.s.), irrespective of under which of the models $\mathbf{y}$ is observed. Note also that the assumption of a common dispersion matrix in $\mathscr{M}_a(\mathbf{V})$ and $\mathscr{M}(\mathbf{V})$ is reasonable, as $\boldsymbol{\delta}$ is considered fixed throughout the paper. In a model with random nuisance effects this assumption would no longer be justified, in general.

It should be observed that the starting point for comparing $\mathscr{M}_a(\mathbf{V})$ with $\mathscr{M}(\mathbf{V})$ is not necessarily the problem of choosing between the models, or between the BLUEs computed under the respective models. In many experimental situations there may be strong experimental evidence (e.g. from previous experiments) supporting the inclusion of nuisance effects in the model. Evaluating the performance of the model $\mathscr{M}_a(\mathbf{V})$ in such a situation, it is, however, rather natural to make comparisons with the reduced model $\mathscr{M}(\mathbf{V})$, as this will reveal the consequences of the presence of nuisance parameters in the model.

On the other hand, consider an experiment where there is uncertainty about the inclusion of nuisance effects in the model, and where the "true" model for the experiment may be assumed to be given by $\mathscr{M}_a(\mathbf{V})$ or $\mathscr{M}(\mathbf{V})$. Adopting the model $\mathscr{M}(\mathbf{V})$ as a parsimonious choice, when $\mathscr{M}_a(\mathbf{V})$ is the "true" model, is known to bias the BLUEs computed under $\mathscr{M}(\mathbf{V})$ [see, e.g., Seber (1977, p. 141)]. Instead of choosing the model $\mathscr{M}(\mathbf{V})$, one could alternatively adopt a cautious strategy of "playing it safe", and accordingly choose the model $\mathscr{M}_a(\mathbf{V})$. If $\mathscr{M}(\mathbf{V})$ then turns out to be the "true" model, the BLUEs computed under $\mathscr{M}_a(\mathbf{V})$ remain unbiased, whereas the variances are inflated [Seber (1977, p. 143)]. In either case, results on the comparison of $\mathscr{M}_a(\mathbf{V})$ and $\mathscr{M}(\mathbf{V})$ may be used to obtain subsets of functionals $\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a$ which are robust against under- or overspecification of the model.

The purpose of this paper is threefold. First we compare the models $\mathscr{M}_a(\mathbf{V})$ and $\mathscr{M}(\mathbf{V})$ w.r.t. BLUEs of functionals $\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a$ (robustness against the presence of nuisance parameters). Then we make a similar comparison of the models $\mathscr{M}(\mathbf{I})$ and $\mathscr{M}(\mathbf{V})$ as well as of their augmented counterparts $\mathscr{M}_a(\mathbf{I})$ and $\mathscr{M}_a(\mathbf{V})$ (dispersion-matrix robustness), and finally we investigate simultaneous robustness against nuisance parameters and an alternative dispersion matrix $\mathbf{V}$.

The paper is organized as follows. Section 2 consists of miscellaneous results on estimability, BLUEs, ranks of matrices, subspace relations, and (generalized) projectors, which will be used throughout the paper.

Section 3 deals with the robustness against nuisance parameters, and contains various alternative characterizations as well as dimensional expressions for the subspace $\mathscr{E}_0(\mathbf{V})$ of functionals $\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a$, which can be estimated with full efficiency under $\mathscr{M}_a(\mathbf{V})$. Several necessary and sufficient conditions for $\mathscr{E}_0(\mathbf{V}) = \mathscr{E}_a$ are, moreover, derived. Many of the results obtained in this section appear to be new even for the case $\mathbf{V} = \mathbf{I}$.

In Section 4 subspaces of $\mathscr{E}_a$ are found wherein the ordinary least-squares estimator (OLSE) continues to be BLUE under an alternative dispersion matrix $\mathbf{V}$. Combining these results with those given in Section 3, a particular subspace of $\mathscr{E}_a$ is isolated within which the OLSE is robust against both the presence of nuisance parameters and an alternative dispersion matrix. A necessary and sufficient condition for the OLSE under $\mathscr{M}_a(\mathbf{I})$ to be BLUE under $\mathscr{M}_a(\mathbf{V})$ for every $\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a$ is obtained as a bonus.

2. PRELIMINARIES

Let $\mathscr{E}$ and $\mathscr{E}_a$ denote the sets of all linear functionals of $\boldsymbol{\gamma}$ which are estimable under the models $\mathscr{M}(\mathbf{V})$ and $\mathscr{M}_a(\mathbf{V})$, respectively. Observe that both $\mathscr{E}$ and $\mathscr{E}_a$ are independent of the dispersion matrix of the underlying model. It is well known that the set $\mathscr{E}$ may be represented in the forms

$$\mathscr{E} = \{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p} = \mathbf{W}'\mathbf{q},\ \mathbf{q} \in \mathbb{R}^n\} = \{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p} \in \mathscr{R}(\mathbf{W}')\}. \tag{2.1}$$

The set $\mathscr{E}_a$ may correspondingly be represented as

$$\mathscr{E}_a = \{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p} = \mathbf{W}'\mathbf{q},\ \mathbf{q} \in \mathscr{R}^{\perp}(\mathbf{Z})\} = \{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p} \in \mathscr{R}(\mathbf{W}'\mathbf{Q}_\mathbf{Z})\}; \tag{2.2}$$

cf. Rao (1946, Theorem 2a), Pringle and Rayner (1971, Theorem 6.12), or Fellman (1976, Lemma 2.1). Note that $\mathscr{E}$ and $\mathscr{E}_a$ are subspaces, since they may be identified with $\mathscr{R}(\mathbf{W}')$ and $\mathscr{R}(\mathbf{W}'\mathbf{Q}_\mathbf{Z})$, respectively. All subsets of $\mathscr{E}$ and $\mathscr{E}_a$, which are constructed in the sequel, are similarly seen to possess the structure of a subspace.

Comparing (2.2) with (2.1), it is plain that $\mathscr{E}_a \subset \mathscr{E}$; moreover,

$$\mathscr{E}_a = \mathscr{E} \quad \Leftrightarrow \quad \mathscr{R}(\mathbf{W}) \cap \mathscr{R}(\mathbf{Z}) = \{\mathbf{0}\}, \tag{2.3}$$

in view of the elementary relation

$$\mathrm{r}(\mathbf{A}) - \mathrm{r}(\mathbf{AB}) = \dim \mathscr{R}(\mathbf{A}') \cap \mathscr{R}^{\perp}(\mathbf{B}), \tag{2.4}$$

holding for arbitrary conformable matrices $\mathbf{A}$ and $\mathbf{B}$ [see, e.g., Marsaglia and Styan (1974, Corollary 6.2)]. To avoid trivialities, we assume throughout the paper that $\mathscr{R}(\mathbf{W}) \not\subset \mathscr{R}(\mathbf{Z})$, so that $\mathscr{E}_a \neq \{\mathbf{0}\}$.

Let $\mathbf{U}'\gamma$ be an arbitrary set of functionals belonging to $\mathscr{E}_a$, and let $\mathbf{U}'\hat{\gamma}$ and $\mathbf{U}'\hat{\gamma}_a$ denote the BLUEs of $\mathbf{U}'\gamma$ computed under $\mathscr{M}(\mathbf{V})$ and $\mathscr{M}_a(\mathbf{V})$, respectively. Under the assumption (1.1), $\mathbf{U}'\hat{\gamma}$ and its dispersion matrix are given by [cf. Mitra and Rao (1968, p. 286)]

$$\mathbf{U}'\hat{\gamma} = \mathbf{U}'\mathbf{C}^- \mathbf{W}'\mathbf{V}^- \mathbf{y}, \qquad D(\mathbf{U}'\hat{\gamma}) = \sigma^2 \mathbf{U}'\mathbf{C}^- \mathbf{U}, \tag{2.5}$$

where $\mathbf{C} = \mathbf{W}'\mathbf{V}^- \mathbf{W}$ is the moment matrix of the model $\mathscr{M}(\mathbf{V})$, appearing also as the matrix in the top left-hand corner of the moment matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{W}'\mathbf{V}^- \mathbf{W} & \mathbf{W}'\mathbf{V}^- \mathbf{Z} \\ \mathbf{Z}'\mathbf{V}^- \mathbf{W} & \mathbf{Z}'\mathbf{V}^- \mathbf{Z} \end{pmatrix}, \tag{2.6}$$

corresponding to the model $\mathscr{M}_a(\mathbf{V})$. Under $\mathscr{M}_a(\mathbf{V})$ one obtains similarly

$$\mathbf{U}'\hat{\gamma}_a = (\mathbf{U}' : \mathbf{0})\mathbf{M}^-(\mathbf{W} : \mathbf{Z})'\mathbf{V}^- \mathbf{y}, \qquad D(\mathbf{U}'\hat{\gamma}_a) = \sigma^2 (\mathbf{U}' : \mathbf{0})\mathbf{M}^-(\mathbf{U}' : \mathbf{0})'. \tag{2.7}$$

Using Rohde's formulae for a generalized inverse of a partitioned n.n.d. matrix [see, e.g., Pringle and Rayner (1971, Section 3.3)], the dispersion matrix of $\mathbf{U}'\hat{\gamma}_a$ may be given in the alternative forms

$$D(\mathbf{U}'\hat{\gamma}_a) = \sigma^2 \mathbf{U}'\left[\mathbf{C}^- + \mathbf{C}^- \mathbf{W}'\mathbf{V}^- \mathbf{Z}(\mathbf{M}/\mathbf{C})^- \mathbf{Z}'\mathbf{V}^- \mathbf{W}\mathbf{C}^-\right]\mathbf{U} \tag{2.8}$$

and

$$D(\mathbf{U}'\hat{\gamma}_a) = \sigma^2 \mathbf{U}'\mathbf{C}_a^- \mathbf{U}, \tag{2.9}$$

where $\mathbf{M}/\mathbf{C}$ denotes the (generalized) Schur complement of $\mathbf{C}$ in $\mathbf{M}$, and $\mathbf{C}_a = \mathbf{M}/\mathbf{Z}'\mathbf{V}^- \mathbf{Z}$.

Due to invariance, arbitrary generalized inverses may be used everywhere in (2.5)–(2.9), and throughout the paper, unless otherwise stated. Such invariances, as well as simplifications of expressions involving generalized inverses, follow, in general, immediately from Lemma 2.2.4 and Lemma 2.2.6

in Rao and Mitra (1971), which will be used tacitly in the sequel. Note, in particular, that

$$\mathscr{R}(\mathbf{W}'\mathbf{V}^{-}\mathbf{W}) = \mathscr{R}(\mathbf{W}'), \tag{2.10}$$

under the assumption (1.1); cf. Rao (1973a, p. 77).

Comparing (2.8) with (2.5) it is seen that $D(\mathbf{U}'\hat{\gamma}_a) \geqslant_{\mathrm{L}} D(\mathbf{U}'\hat{\gamma})$, implying

$$\mathrm{Var}(\mathbf{p}'\hat{\gamma}_a) \geqslant \mathrm{Var}(\mathbf{p}'\hat{\gamma}) \qquad \text{for every} \quad \mathbf{p}'\gamma \in \mathscr{E}_a, \tag{2.11}$$

which, combined with $\mathscr{E}_a \subset \mathscr{E}$, shows that the model $\mathscr{M}(\mathbf{V})$ dominates (or is at least as good as) the model $\mathscr{M}_a(\mathbf{V})$ in the sense of Ehrenfeld (1955b, p. 59) and Kiefer (1959, p. 286). A comparison of (2.8) and (2.5) shows further that

$$D(\mathbf{U}'\hat{\gamma}_a) = D(\mathbf{U}'\hat{\gamma}) \quad \Leftrightarrow \quad \mathbf{U}'\mathbf{C}^{-}\mathbf{W}'\mathbf{V}^{-}\mathbf{Z} = \mathbf{0}, \tag{2.12}$$

as $(\mathbf{M}/\mathbf{C})^{-}$ may be chosen positive definite.

The expressions for the BLUEs, given in (2.5) and (2.7), are obtained through minimization and resulting normal equations, and have, consequently, a distinctively algebraic flavor. Another approach is to express the BLUEs in terms of (generalized) projectors, an approach put forth, in the case of a singular $\mathbf{V}$, by Stein (1972) and by Rao (1973b, 1974). Below we state some basic properties of such projectors that will be needed later on, and which correspond to the particular case of a $\mathbf{V}$ satisfying (1.1).

Given $\mathbf{A} \in \mathbb{R}^{p \times q}$ and an n.n.d. $\mathbf{S} \in \mathbb{R}^{p \times p}$ such that $\mathscr{R}(\mathbf{A}) \subset \mathscr{R}(\mathbf{S})$, we have the following direct-sum decomposition:

$$\mathscr{R}(\mathbf{S}) = \mathscr{R}(\mathbf{A}) \oplus \mathscr{R}(\mathbf{S}\mathbf{A}^{\perp}) \tag{2.13}$$

[cf. Rosenberg (1969, Lemma 2.3) or Rao (1974, Lemma 2.1)], and hence one may consider the projector $\mathbf{P}_{\mathbf{A}|\mathbf{S}\mathbf{A}^{\perp}}$, projecting vectors in $\mathscr{R}(\mathbf{S})$ onto $\mathscr{R}(\mathbf{A})$ along $\mathscr{R}(\mathbf{S}\mathbf{A}^{\perp})$. As pointed out by Rao (1974, p. 444), such a projector $\mathbf{P}_{\mathbf{A}|\mathbf{S}\mathbf{A}^{\perp}}$ is, in general, neither unique nor idempotent, since it may be extended arbitrarily from $\mathscr{R}(\mathbf{S})$ to $\mathbb{R}^{p}$. A general representation of all such projectors $\mathbf{P}_{\mathbf{A}|\mathbf{S}\mathbf{A}^{\perp}}$ is given by

$$\mathbf{A}(\mathbf{A}'\mathbf{S}^{-}\mathbf{A})^{-}\mathbf{A}'\mathbf{S}^{-} + \mathbf{F}(\mathbf{I} - \mathbf{S}\mathbf{S}^{-}), \tag{2.14}$$

where $\mathbf{F}$ is arbitrary; see, e.g., Rao and Yanai (1979, Theorem 8). It is useful to observe that $\mathscr{R}(\mathbf{S}\mathbf{A}^{\perp})$ may be written in the alternative form

$$\mathscr{R}(\mathbf{S}\mathbf{A}^{\perp}) = \mathscr{R}^{\perp}(\mathbf{S}^{-}\mathbf{A}) \cap \mathscr{R}(\mathbf{S}), \tag{2.15}$$

in view of the equality

$$\mathscr{R}^{\perp}(\mathbf{S}\mathbf{A}^{\perp}) = \mathscr{R}(\mathbf{S}^{-}\mathbf{A} : \mathbf{I} - \mathbf{S}^{-}\mathbf{S}); \tag{2.16}$$

cf. (2.5) in Rao (1973b). Note also that, in the particular case $\mathbf{S} = \mathbf{I}$, the projector $\mathbf{P}_{\mathbf{A}|\mathbf{S}\mathbf{A}^{\perp}}$ reduces to the usual orthogonal projector $\mathbf{P}_{\mathbf{A}}$.

The BLUEs $\mathbf{U}'\hat{\gamma}$ and $\mathbf{U}'\hat{\gamma}_a$ may now be expressed in terms of such (generalized) projectors. From (2.2) it follows that $\mathbf{U}'\gamma = \mathbf{K}'\mathbf{W}\gamma$ for some $\mathbf{K}$ such that $\mathscr{R}(\mathbf{K}) \subset \mathscr{R}^{\perp}(\mathbf{Z})$, and hence

$$\mathbf{U}'\hat{\gamma} = \mathbf{K}'\mathbf{P}_{\mathbf{W}|\mathbf{V}\mathbf{W}^{\perp}}\mathbf{y} \tag{2.17}$$

and

$$\mathbf{U}'\hat{\gamma}_a = \mathbf{K}'\mathbf{P}_{(\mathbf{W}:\mathbf{Z})|\mathbf{V}(\mathbf{W}:\mathbf{Z})^{\perp}}\mathbf{y}. \tag{2.18}$$

The following lemma extends a result, given by Shinozaki and Sibuya (1974, Lemma 1) for idempotent projectors, to the more general class of projectors considered above. The proof is straightforward, and follows the proof given by Shinozaki and Sibuya, with obvious modifications.

LEMMA 2.1. *Let* $\mathbf{A} \in \mathbb{R}^{p\times q}$, $\mathbf{B} \in \mathbb{R}^{p\times r}$, *and let* $\mathbf{S} \in \mathbb{R}^{p\times p}$ *be n.n.d. such that* $\mathscr{R}(\mathbf{A}:\mathbf{B}) \subset \mathscr{R}(\mathbf{S})$. *Then*

$$\mathscr{R}(\mathbf{P}_{\mathbf{A}|\mathbf{S}\mathbf{A}^{\perp}}\mathbf{B}) = \mathscr{R}(\mathbf{A}) \cap \left[\mathscr{R}(\mathbf{S}\mathbf{A}^{\perp}) + \mathscr{R}(\mathbf{B})\right] \tag{2.19}$$

for every choice of the projector $\mathbf{P}_{\mathbf{A}|\mathbf{S}\mathbf{A}^{\perp}}$.

Although we shall not make explicit use of this fact, it may be of interest to note that [cf. Rao (1974, Lemma 2.10)]

$$\mathbf{P}_{\mathbf{A}|\mathbf{S}\mathbf{A}^{\perp}}\mathbf{S} = \mathbf{P}_{\mathbf{A}(\mathbf{S}^{-})}\mathbf{S},$$

where $\mathbf{P}_{\mathbf{A}(\mathbf{S}^{-})}$ denotes a projector into $\mathscr{R}(\mathbf{A})$ w.r.t. the seminorm defined by an n.n.d. generalized inverse $\mathbf{S}^{-}$ of $\mathbf{S}$, as considered in detail by Mitra and Rao (1974).

A final result that will be used extensively in the sequel is the modular law for subspaces stating that, given subspaces $\mathscr{U}$, $\mathscr{V}$, and $\mathscr{W}$ of $\mathbb{R}^p$, we have

$$\mathscr{U} \supset \mathscr{W} \quad \Rightarrow \quad \mathscr{U} \cap (\mathscr{V} + \mathscr{W}) = (\mathscr{U} \cap \mathscr{V}) + \mathscr{W}; \tag{2.20}$$

see, e.g., Nordström and von Rosen (1987, Lemma 2.1).

3. CHARACTERIZATIONS OF EFFICIENTLY ESTIMABLE FUNCTIONALS

Let $\mathscr{E}_0(\mathbf{V})$ denote the subset of $\mathscr{E}_a$ consisting of all those functionals of $\boldsymbol{\gamma}$ for which the BLUE under $\mathscr{M}_a(\mathbf{V})$ possesses the same variance as the BLUE under $\mathscr{M}(\mathbf{V})$, i.e.,

$$\mathscr{E}_0(\mathbf{V}) = \{\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a : \mathrm{Var}(\mathbf{p}'\hat{\boldsymbol{\gamma}}_a) = \mathrm{Var}(\mathbf{p}'\hat{\boldsymbol{\gamma}})\}. \tag{3.1}$$

The set $\mathscr{E}_0(\mathbf{V})$ comprises hence the parametric functionals of the main parameters which are estimable with full efficiency under $\mathscr{M}_a(\mathbf{V})$. From (2.12) it is seen that $\mathscr{E}_0(\mathbf{V})$ is a subspace of $\mathscr{E}_a$. When $\mathbf{V} = \mathbf{I}$, the corresponding subspace is denoted by $\mathscr{E}_0(\mathbf{I})$, to indicate the general dependence on the assumed dispersion matrix of the underlying models.

THEOREM 3.1. *The subspace $\mathscr{E}_0(\mathbf{V})$ is given by*

$$\mathscr{E}_0(\mathbf{V}) = \{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p} = \mathbf{W}'\mathbf{q},\, \mathbf{q} \in \mathscr{R}^{\perp}(\mathbf{V}\mathbf{W}^{\perp}) \cap \mathscr{R}(\mathbf{V}) \cap \mathscr{R}^{\perp}(\mathbf{Z})\} \tag{3.2}$$

$$= \{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p} \in \mathscr{R}(\mathbf{W}'\mathbf{V}^{-}\mathbf{W}\mathbf{Q}_{\mathbf{W}'\mathbf{V}^{-}\mathbf{Z}})\}. \tag{3.3}$$

Proof. On account of (2.2), (2.12), and (2.14), the set (3.1) is seen to be representable in the form

$$\left\{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p} = \mathbf{W}'\mathbf{q},\, \mathbf{q} \in \mathscr{R}^{\perp}(\mathbf{Z}) \quad \text{and} \quad \mathbf{q}'\mathbf{P}_{\mathbf{W}|\mathbf{V}\mathbf{W}^{\perp}}\mathbf{Z} = \mathbf{0}\right\},$$

and hence $\mathbf{q}$ must be chosen from the subspace

$$\mathscr{R}^{\perp}(\mathbf{Z}) \cap \{\mathscr{R}^{\perp}(\mathbf{W}) + [\mathscr{R}^{\perp}(\mathbf{V}\mathbf{W}^{\perp}) \cap \mathscr{R}^{\perp}(\mathbf{Z})]\}, \tag{3.4}$$

in view of Lemma 2.1. On the other hand, it follows from (1.1) and (2.16) that $\mathscr{N}(\mathbf{V}) \subset \mathscr{R}^{\perp}(\mathbf{V}\mathbf{W}^{\perp}) \cap \mathscr{R}^{\perp}(\mathbf{Z})$, yielding the decomposition

$$\mathscr{R}^{\perp}(\mathbf{V}\mathbf{W}^{\perp}) \cap \mathscr{R}^{\perp}(\mathbf{Z}) = \mathscr{N}(\mathbf{V}) \boxplus [\mathscr{R}^{\perp}(\mathbf{V}\mathbf{W}^{\perp}) \cap \mathscr{R}(\mathbf{V}) \cap \mathscr{R}^{\perp}(\mathbf{Z})],$$

from which it is seen that (3.4) coincides with

$$\mathscr{R}^{\perp}(\mathbf{Z}) \cap \{\mathscr{R}^{\perp}(\mathbf{W}) + [\mathscr{R}^{\perp}(\mathbf{V}\mathbf{W}^{\perp}) \cap \mathscr{R}(\mathbf{V}) \cap \mathscr{R}^{\perp}(\mathbf{Z})]\}, \tag{3.5}$$

in view of (1.1). Applying (2.20) to (3.5), and observing that any component of $\mathbf{q}$ lying in $\mathscr{R}^{\perp}(\mathbf{Z}) \cap \mathscr{R}^{\perp}(\mathbf{W})$ is nullified when premultiplied by $\mathbf{W}'$, yields the representation (3.2). The alternative representation (3.3) is obtained straightforwardly from (3.2) using (2.10). ■

COROLLARY 3.1. *The subspace* $\mathscr{E}_0(\mathbf{I})$ *is given by*

$$\mathscr{E}_0(\mathbf{I}) = \{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p} = \mathbf{W}'\mathbf{q},\ \mathbf{q} \in \mathscr{R}(\mathbf{W}) \cap \mathscr{R}^{\perp}(\mathbf{Z})\} \tag{3.6}$$

$$= \{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p} \in \mathscr{R}(\mathbf{W}'\mathbf{W}\mathbf{Q}_{\mathbf{W}'\mathbf{Z}})\}. \tag{3.7}$$

The representation (3.6) is due to Fellman (1985, Theorem 3).

Referring to the problem of under- versus overspecified models, touched upon in the introductory section, we have the following property for functionals belonging to $\mathscr{E}_0(\mathbf{V})$:

THEOREM 3.2. *Let* $\mathrm{LUE}_a(\mathbf{p}'\boldsymbol{\gamma})$ *denote the class of linear unbiased estimators of* $\mathbf{p}'\boldsymbol{\gamma}$ *under the model* $\mathscr{M}_a(\mathbf{V})$. *Then*

$$\mathscr{E}_0(\mathbf{V}) = \{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p}'\hat{\boldsymbol{\gamma}} \in \mathrm{LUE}_a(\mathbf{p}'\boldsymbol{\gamma})\}, \tag{3.8}$$

i.e., the subspace $\mathscr{E}_0(\mathbf{V})$ *comprises precisely those functionals* $\mathbf{p}'\boldsymbol{\gamma}$ *whose* BLUE *under* $\mathscr{M}(\mathbf{V})$ *remains unbiased under* $\mathscr{M}_a(\mathbf{V})$.

Proof. Since $\mathbf{p}'\hat{\boldsymbol{\gamma}} = \mathbf{p}'\mathbf{C}^-\mathbf{W}'\mathbf{V}^-\mathbf{y}$ [cf. (2.5)], we have

$$\begin{aligned}
\mathbf{p}'\hat{\boldsymbol{\gamma}} \in \mathrm{LUE}_a(\mathbf{p}'\boldsymbol{\gamma}) \quad &\Leftrightarrow \quad E_a(\mathbf{p}'\hat{\boldsymbol{\gamma}}) \equiv \mathbf{p}'\boldsymbol{\gamma} \\
&\Leftrightarrow \quad \mathbf{p}'\mathbf{C}^-\mathbf{C}\boldsymbol{\gamma} + \mathbf{p}'\mathbf{C}^-\mathbf{W}'\mathbf{V}^-\mathbf{Z}\boldsymbol{\delta} \equiv \mathbf{p}'\boldsymbol{\gamma} \\
&\Leftrightarrow \quad \mathbf{p}'\mathbf{C}^-\mathbf{C} = \mathbf{p}' \text{ and } \mathbf{p}'\mathbf{C}^-\mathbf{W}'\mathbf{V}^-\mathbf{Z} = \mathbf{0} \\
&\Leftrightarrow \quad \mathbf{p} = \mathbf{C}\mathbf{s} \text{ for some } \mathbf{s} \text{ and } \mathbf{s}'\mathbf{W}'\mathbf{V}^-\mathbf{Z} = \mathbf{0} \\
&\Leftrightarrow \quad \mathbf{p} \in \mathscr{R}(\mathbf{W}'\mathbf{V}^-\mathbf{W}\mathbf{Q}_{\mathbf{W}'\mathbf{V}^-\mathbf{Z}}) \\
&\Leftrightarrow \quad \mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_0(\mathbf{V}),
\end{aligned}$$

where the last equivalence follows by applying (3.3). ■

It is interesting to compare the characterization (3.8) of $\mathscr{E}_0(\mathbf{V})$ with the definition (3.1). It transpires that, irrespective of whether one is looking for the subset of functionals $\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a$ that are "bias-robust" w.r.t. underspecification of the model [i.e., whose BLUEs under $\mathscr{M}(\mathbf{V})$ remain unbiased when

$\mathscr{M}_a(\mathbf{V})$ is the true model], or whether one focuses on those $\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a$ that are "variance-robust" w.r.t. overspecification of the model [i.e., whose BLUEs under $\mathscr{M}_a(\mathbf{V})$ retain their variances when $\mathscr{M}(\mathbf{V})$ is the true model], one ends up with the same subspace $\mathscr{E}_0(\mathbf{V})$ of $\mathscr{E}_a$.

The particular case of (3.8), corresponding to $\mathbf{V} = \mathbf{I}$, extends the part "(S10) $\Leftrightarrow$ (S13)" of Theorem 6 in Baksalary (1987), stating (in our notation) that

$$\mathbf{W}'\mathbf{Q_Z}\mathbf{W}\hat{\boldsymbol{\gamma}} \in \mathrm{LUE}_a(\mathbf{W}'\mathbf{Q_Z}\mathbf{W}\boldsymbol{\gamma}) \quad \Leftrightarrow \quad \mathscr{E}_0(\mathbf{I}) = \mathscr{E}_a,$$

which, on account of (2.2), is obviously equivalent to

$$\mathbf{p}'\hat{\boldsymbol{\gamma}} \in \mathrm{LUE}_a(\mathbf{p}'\boldsymbol{\gamma}) \quad \text{for every } \mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a \quad \Leftrightarrow \quad \mathscr{E}_0(\mathbf{I}) = \mathscr{E}_a.$$

In view of the dominance of $\mathscr{M}(\mathbf{V})$ over $\mathscr{M}_a(\mathbf{V})$, Theorem 3.2 may actually be strengthened to:

THEOREM 3.3. *The subspace $\mathscr{E}_0(\mathbf{V})$ can be characterized as*

$$\mathscr{E}_0(\mathbf{V}) = \{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p}'\hat{\boldsymbol{\gamma}} = \mathbf{p}'\hat{\boldsymbol{\gamma}}_a\}. \tag{3.9}$$

Proof. Let $\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_0(\mathbf{V})$. On account of Theorem 3.2, we have $E_a(\mathbf{p}'\hat{\boldsymbol{\gamma}}) = \mathbf{p}'\boldsymbol{\gamma}$, and since the same dispersion matrix is assumed in $\mathscr{M}(\mathbf{V})$ and $\mathscr{M}_a(\mathbf{V})$, we also have $\mathrm{Var}_a(\mathbf{p}'\hat{\boldsymbol{\gamma}}) = \mathrm{Var}(\mathbf{p}'\hat{\boldsymbol{\gamma}})$. Hence, it follows from (2.11) that

$$\mathrm{Var}_a(\mathbf{p}'\hat{\boldsymbol{\gamma}}) = \mathrm{Var}(\mathbf{p}'\hat{\boldsymbol{\gamma}}) \leqslant \mathrm{Var}_a(\mathbf{p}'\hat{\boldsymbol{\gamma}}_a). \tag{3.10}$$

Consequently, equality must hold in (3.10), and $\mathbf{p}'\hat{\boldsymbol{\gamma}}$ is BLUE of $\mathbf{p}'\boldsymbol{\gamma}$ under $\mathscr{M}_a(\mathbf{V})$. Assume, conversely, that $\mathbf{p}'\hat{\boldsymbol{\gamma}}$ is BLUE of $\mathbf{p}'\boldsymbol{\gamma}$ under $\mathscr{M}_a(\mathbf{V})$. Then $\mathbf{p}'\hat{\boldsymbol{\gamma}} \in \mathrm{LUE}_a(\mathbf{p}'\boldsymbol{\gamma})$, and thus $\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_0(\mathbf{V})$, in view of Theorem 3.2. ■

The dimension of the subspace $\mathscr{E}_0(\mathbf{V})$ is given in the following

THEOREM 3.4. *The dimension of $\mathscr{E}_0(\mathbf{V})$ is obtained as*

$$\dim \mathscr{E}_0(\mathbf{V}) = \dim \mathscr{R}^{\perp}(\mathbf{V}\mathbf{W}^{\perp}) \cap \mathscr{R}(\mathbf{V}) \cap \mathscr{R}^{\perp}(\mathbf{Z}) \tag{3.11}$$

$$= \dim \mathscr{R}(\mathbf{W}) \cap \mathscr{R}(\mathbf{V}\mathbf{Z}^{\perp}) \tag{3.12}$$

$$= \dim \mathscr{E}_a - [\mathrm{r}(\mathbf{W}'\mathbf{V}^{-}\mathbf{Z}) - \dim \mathscr{R}(\mathbf{W}) \cap \mathscr{R}(\mathbf{Z})] \tag{3.13}$$

$$= \mathrm{r}(\mathbf{W}) - \mathrm{r}(\mathbf{W}'\mathbf{V}^{-}\mathbf{Z}). \tag{3.14}$$

Proof. The equality in (3.11) follows from (3.2), using (2.4) and (2.13). The equality (3.14) is similarly obtained from (3.3), in view of (2.4) and (2.10). On the other hand, applying (2.4) to (3.14) yields

$$\dim \mathscr{E}_0(\mathbf{V}) = \dim \mathscr{R}(\mathbf{W}) \cap \mathscr{R}^{\perp}(\mathbf{V}^{-}\mathbf{Z}),$$

or equivalently,

$$\dim \mathscr{E}_0(\mathbf{V}) = \dim \mathscr{R}(\mathbf{W}) \cap \mathscr{R}^{\perp}(\mathbf{V}^{-}\mathbf{Z}) \cap \mathscr{R}(\mathbf{V}),$$

which, combined with (2.15), establishes (3.12). Observing that

$$\dim \mathscr{E} - \dim \mathscr{E}_0(\mathbf{V}) = \mathrm{r}(\mathbf{W}'\mathbf{V}^{-}\mathbf{Z}), \tag{3.15}$$

in view of (3.14), and observing further that

$$\dim \mathscr{E} - \dim \mathscr{E}_a = \dim \mathscr{R}(\mathbf{W}) \cap \mathscr{R}(\mathbf{Z}),$$

on account of (2.1), (2.2), and (2.4), the equality (3.13) follows upon rewriting $\dim \mathscr{E}_0(\mathbf{V})$ as

$$\dim \mathscr{E}_0(\mathbf{V}) = \dim \mathscr{E}_a - \left[\dim \mathscr{E} - \dim \mathscr{E}_0(\mathbf{V}) - (\dim \mathscr{E} - \dim \mathscr{E}_a)\right]. \qquad \blacksquare$$

COROLLARY 3.2. *The dimension of $\mathscr{E}_0(\mathbf{I})$ is obtained as*

$$\dim \mathscr{E}_0(\mathbf{I}) = \dim \mathscr{R}(\mathbf{W}) \cap \mathscr{R}^{\perp}(\mathbf{Z}) \tag{3.16}$$

$$= \dim \mathscr{E}_a - \left[\mathrm{r}(\mathbf{W}'\mathbf{Z}) - \dim \mathscr{R}(\mathbf{W}) \cap \mathscr{R}(\mathbf{Z})\right] \tag{3.17}$$

$$= \mathrm{r}(\mathbf{W}) - \mathrm{r}(\mathbf{W}'\mathbf{Z}). \tag{3.18}$$

The equality (3.16) was established by Fellman (1985, Theorem 4), and (3.18) has been obtained independently by Nordström and Fellman (1988, Theorem 4.3), and Baksalary (1989, Theorem).

Comparing (3.16) with (3.6), and (3.11) with (3.2), it is seen that the representations (3.2) and (3.6) are minimal in that the vectors $\mathbf{q}$ and $\mathbf{p}$ are in 1-1 correspondence, i.e., no redundancy is included in the space of permissible $\mathbf{q}$'s. In the sequel, all sets of this type are given in such a minimal form.

Motivated by Fisher's inequality for combinatorially balanced incomplete block designs, Baksalary (1989, Definition 2) defined a linear model $\mathscr{M}_a(\mathbf{I})$ to satisfy Fisher's condition if $r(W'Z) = r(W)$, or equivalently, $\mathscr{E}_0(\mathbf{I}) = \{\mathbf{0}\}$ holds. In view of (3.14), this definition can be extended to the model $\mathscr{M}_a(\mathbf{V})$, which may correspondingly be said to satisfy Fisher's condition if

$$r(\mathbf{W}'\mathbf{V}^-\mathbf{Z}) = r(\mathbf{W}), \tag{3.19}$$

or equivalently, $\mathscr{E}_0(\mathbf{V}) = \{\mathbf{0}\}$ holds.

The following theorem deals with the other extreme situation in which $\mathscr{E}_0(\mathbf{V}) = \mathscr{E}_a$, and extends, to the model $\mathscr{M}_a(\mathbf{V})$, results given earlier by Ehrenfeld (1955a), Fellman (1976), and Baksalary (1984, 1987) for the model $\mathscr{M}_a(\mathbf{I})$.

THEOREM 3.5. *The subspaces $\mathscr{E}_0(\mathbf{V})$ and $\mathscr{E}_a$ coincide if and only if any of the following equivalent conditions holds*:

$$\mathbf{P}_{\mathbf{W}|\mathbf{VW}^\perp}\mathbf{P}_{\mathbf{Z}|\mathbf{VZ}^\perp} = \left(\mathbf{P}_{\mathbf{W}|\mathbf{VW}^\perp}\mathbf{P}_{\mathbf{Z}|\mathbf{VZ}^\perp}\right)^2, \tag{3.20}$$

$$\mathbf{W}'\mathbf{V}^-\mathbf{Z} = \mathbf{W}'\mathbf{V}^-\mathbf{P}_{\mathbf{Z}|\mathbf{VZ}^\perp}\mathbf{P}_{\mathbf{W}|\mathbf{VW}^\perp}\mathbf{Z}, \tag{3.21}$$

$$r(\mathbf{W}'\mathbf{V}^-\mathbf{Z}) = \dim \mathscr{R}(\mathbf{W}) \cap \mathscr{R}(\mathbf{Z}), \tag{3.22}$$

$$\mathbf{C} \geqslant_{\mathrm{rs}} \mathbf{C}_a, \tag{3.23}$$

$$\mathbf{W}'\mathbf{V}^-\mathbf{W} \geqslant_{\mathrm{rs}} \mathbf{W}'\mathbf{V}^-\mathbf{Z}(\mathbf{Z}'\mathbf{V}^-\mathbf{Z})^-\mathbf{Z}'\mathbf{V}^-\mathbf{W}. \tag{3.24}$$

In the condition (3.20) *the projectors* $\mathbf{P}_{\mathbf{W}|\mathbf{VW}^\perp}$ *and* $\mathbf{P}_{\mathbf{Z}|\mathbf{VZ}^\perp}$ *may be chosen arbitrarily with the exception of the two projectors* $\mathbf{P}_{\mathbf{Z}|\mathbf{VZ}^\perp}$ *at the end of both sides of the equality, which must be chosen the same.*

Proof. Note first that $\mathscr{E}_0(\mathbf{V}) = \mathscr{E}_a$ if and only if

$$D(\mathbf{C}_a\hat{\gamma}_a) = D(\mathbf{C}_a\hat{\gamma}), \tag{3.25}$$

as $\mathscr{R}(\mathbf{C}_a) = \mathscr{R}(\mathbf{W}'\mathbf{Q}_\mathbf{Z})$; cf. Stepniak, Wang, and Wu (1984, p. 363). From (2.12) it then follows that (3.25) is equivalent to $\mathbf{C}_a\mathbf{C}^-\mathbf{W}'\mathbf{V}^-\mathbf{Z} = \mathbf{0}$, which, upon simplification, establishes the condition (3.21). Pre- and postmultiplying both sides of (3.21) by $\mathbf{WC}^-$ and $(\mathbf{Z}'\mathbf{V}^-\mathbf{Z})^-\mathbf{Z}'\mathbf{V}^-$, respectively, yields (3.20), which gives back (3.21), when pre- and postmultiplied by $\mathbf{W}'\mathbf{V}^-$ and $\mathbf{Z}$, respectively. The condition (3.20) is hence established in the stated generality by checking that, under the assumption (1.1), the products $\mathbf{P}_{\mathbf{W}|\mathbf{VW}^\perp}\mathbf{P}_{\mathbf{Z}|\mathbf{VZ}^\perp}$

and $\mathbf{P}_{\mathbf{Z}|\mathbf{VZ}^{\perp}}\mathbf{P}_{\mathbf{W}|\mathbf{VW}^{\perp}}$ are unique w.r.t. the choice of $\mathbf{P}_{\mathbf{W}|\mathbf{VW}^{\perp}}$ and $\mathbf{P}_{\mathbf{Z}|\mathbf{VZ}^{\perp}}$, respectively; cf. (2.14). It remains to prove (3.23) and (3.24), since (3.22) is a direct consequence of (3.13). For this observe that, using (3.25), (2.9), and (2.5), $\mathscr{E}_0(\mathbf{V}) = \mathscr{E}_a$ if and only if

$$\mathbf{C}_a(\mathbf{C}_a^- - \mathbf{C}^-)\mathbf{C}_a = \mathbf{0},$$

which, combined with the rank equality

$$\mathrm{r}[\mathbf{C}_a(\mathbf{C}_a^- - \mathbf{C}^-)\mathbf{C}_a] = \mathrm{r}(\mathbf{C} - \mathbf{C}_a) - [\mathrm{r}(\mathbf{C}) - \mathrm{r}(\mathbf{C}_a)],$$

given by Styan (1985, p. 48) [see also (2.8) in Baksalary, Nordström, and Styan (1989)], yields

$$\mathrm{r}(\mathbf{C} - \mathbf{C}_a) = \mathrm{r}(\mathbf{C}) - \mathrm{r}(\mathbf{C}_a), \tag{3.26}$$

i.e., the condition (3.23). Rewriting (3.26) in the form

$$\mathrm{r}(\mathbf{C}_a) = \mathrm{r}(\mathbf{C}) - \mathrm{r}(\mathbf{C} - \mathbf{C}_a),$$

and writing out the matrices $\mathbf{C}$ and $\mathbf{C}_a$, establishes the condition (3.24). ∎

COROLLARY 3.3. *The subspaces $\mathscr{E}_0(\mathbf{I})$ and $\mathscr{E}_a$ coincide if and only if any of the following equivalent conditions holds*:

$$\mathbf{P}_{\mathbf{W}}\mathbf{P}_{\mathbf{Z}} = \mathbf{P}_{\mathbf{Z}}\mathbf{P}_{\mathbf{W}}, \tag{3.27}$$

$$\mathbf{W}'\mathbf{Z} = \mathbf{W}'\mathbf{P}_{\mathbf{Z}}\mathbf{P}_{\mathbf{W}}\mathbf{Z}, \tag{3.28}$$

$$\mathrm{r}(\mathbf{W}'\mathbf{Z}) = \dim \mathscr{R}(\mathbf{W}) \cap \mathscr{R}(\mathbf{Z}), \tag{3.29}$$

$$\mathbf{W}'\mathbf{W} \geqslant_{\mathrm{rs}} \mathbf{W}'\mathbf{Q}_{\mathbf{Z}}\mathbf{W}, \tag{3.30}$$

$$\mathbf{W}'\mathbf{W} \geqslant_{\mathrm{rs}} \mathbf{W}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^-\mathbf{Z}'\mathbf{W}. \tag{3.31}$$

The condition (3.27) was established by Baksalary (1984, Theorem 2.3), and (3.28) is due to Fellman (1976, Theorem 2.3). The condition (3.29) is given as condition (A34) of Theorem 1 in Baksalary (1987) [cf. also (21)–(22) in Nordström and Fellman (1988)], which comprises a comprehensive list of conditions that are equivalent to (3.27).

A comparison of (3.27)–(3.29) with (3.20)–(3.22) clearly indicates the kind of modifications that will have to be made in Baksalary's theorem, when the orthogonal projectors are replaced by (generalized) oblique ones. Using this correspondence, additional conditions for $\mathscr{E}_0(\mathbf{V}) = \mathscr{E}_a$ can be derived in a straightforward manner.

The rank subtractivity conditions (3.23) and (3.24) appear to be new even for the model $\mathscr{M}_a(\mathbf{I})$. It is interesting to note that, since $\mathbf{C}$ and $\mathbf{C}_a$ are n.n.d., we have [see Baksalary, Kala, and Kłaczyński (1983, p. 83) or Hartwig and Styan (1987, Theorem 2.1)]

$$\mathbf{C} \geqslant_{\mathrm{rs}} \mathbf{C}_a \quad \Rightarrow \quad \mathbf{C} \geqslant_{\mathrm{L}} \mathbf{C}_a,$$

the latter condition being the well-known necessary and sufficient condition (trivially fulfilled in the present case) for the dominance of $\mathscr{M}(\mathbf{V})$ over $\mathscr{M}_a(\mathbf{V})$; cf. Stepniak, Wang, and Wu (1984, Corollary 2). Note also that the condition (3.24) is one of rank subtractivity of the matrices in the Schur complement $\mathbf{M}/\mathbf{Z}'\mathbf{V}^-\mathbf{Z}$.

It may be pointed out that in models related to experimental designs such as block or row-column designs, $\mathscr{E}_0(\mathbf{V})$ consists of the subspace of treatment contrasts with efficiency factor equal to unity, under the dispersion structure $\mathbf{V}$. In the following section we obtain, among other things, an algebraic representation and various characterizations of the subspace $\mathscr{E}_0(\mathbf{V}) \cap \mathscr{E}_0(\mathbf{I})$, which, in the case of a design model, corresponds to the treatment contrasts which are estimable with full efficiency both under the usual model of uncorrelated observations and under an alternative model with correlated observations. A systematic application of the results of this paper to various models in experimental design will be given elsewhere [for some applications to block designs in the case $D(\mathbf{y}) = \sigma^2\mathbf{I}$, see Nordström and Fellman (1988, Section 5) and Baksalary (1989)].

4. DISPERSION-MATRIX ROBUSTNESS

The results given in Section 3 deal essentially with the robustness of BLUEs against the presence of nuisance parameters in the model, under the assumption of a known dispersion matrix. In this section we study the robustness of OLSEs, computed under the models $\mathscr{M}(\mathbf{I})$ and $\mathscr{M}_a(\mathbf{I})$, against an alternative dispersion matrix. The results obtained are combined with those in Section 3, in order to isolate the subset of $\mathscr{E}_a$ comprising those functionals $\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a$ for which the OLSE under $\mathscr{M}(\mathbf{I})$ continues to be BLUE under both $\mathscr{M}(\mathbf{V})$ and $\mathscr{M}_a(\mathbf{V})$, i.e., for which the OLSE under $\mathscr{M}(\mathbf{I})$ is robust against both the presence of nuisance parameters $\boldsymbol{\delta}$ and an alternative dispersion matrix $\mathbf{V}$.

Given $\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a$, let $\mathbf{p}'\tilde{\boldsymbol{\gamma}}$ and $\mathbf{p}'\tilde{\boldsymbol{\gamma}}_a$ denote the OLSEs of $\mathbf{p}'\boldsymbol{\gamma}$ computed under $\mathscr{M}(\mathbf{I})$ and $\mathscr{M}_a(\mathbf{I})$, respectively, and assume throughout the rest of this section that the dispersion matrix of the models is given by $D(\mathbf{y}) = \sigma^2\mathbf{V}$, with $\mathbf{V} \neq \mathbf{I}$ and possibly singular, but still satisfying (1.1). Under this assumption, the set $\mathscr{E}_0(\mathbf{I})$ is no longer defined according to the convention in Section 3, but may be identified with the subspace given in (3.6), yielding the following:

THEOREM 4.1. *The subspace $\mathscr{E}_0(\mathbf{I})$ can be characterized as*

$$\mathscr{E}_0(\mathbf{I}) = \{\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a : \mathbf{p}'\tilde{\boldsymbol{\gamma}} = \mathbf{p}'\tilde{\boldsymbol{\gamma}}_a\}. \tag{4.1}$$

Proof. Let $\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a$. From (2.17) and (2.18) it follows that

$$\mathbf{p}'\tilde{\boldsymbol{\gamma}} = \mathbf{q}'\mathbf{P}_{\mathbf{W}}\mathbf{y} \tag{4.2}$$

and

$$\mathbf{p}'\tilde{\boldsymbol{\gamma}}_a = \mathbf{q}'\mathbf{P}_{(\mathbf{W}:\mathbf{Z})}\mathbf{y}, \tag{4.3}$$

where $\mathbf{p} = \mathbf{W}'\mathbf{q}$ and $\mathbf{q} \in \mathscr{R}^{\perp}(\mathbf{Z})$; cf. (2.2). Since $\mathbf{y} \in \mathscr{R}(\mathbf{V})$ (a.s.), it follows from (4.2) and (4.3) that

$$\mathbf{p}'\tilde{\boldsymbol{\gamma}} = \mathbf{p}'\tilde{\boldsymbol{\gamma}}_a \quad \Leftrightarrow \quad \mathbf{q}'(\mathbf{P}_{(\mathbf{W}:\mathbf{Z})} - \mathbf{P}_{\mathbf{W}})\mathbf{V} = \mathbf{0}.$$

Observing that $\mathbf{P}_{(\mathbf{W}:\mathbf{Z})} - \mathbf{P}_{\mathbf{W}}$ is the orthogonal projector onto $\mathscr{R}^{\perp}(\mathbf{W}) \cap \mathscr{R}(\mathbf{W}:\mathbf{Z})$, and using Lemma 2.1 as well as (1.1), the set on the r.h.s. of (4.1) comprises the vectors $\mathbf{p}$ such that $\mathbf{p} = \mathbf{W}'\mathbf{q}$, with $\mathbf{q}$ chosen from the subspace

$$\mathscr{R}^{\perp}(\mathbf{Z}) \cap \{\mathscr{R}(\mathbf{W}) \boxplus [\mathscr{R}^{\perp}(\mathbf{W}) \cap \mathscr{R}^{\perp}(\mathbf{Z})]\}. \tag{4.4}$$

Applying (2.20) to (4.4) yields the minimal representation

$$\{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p} = \mathbf{W}'\mathbf{q},\ \mathbf{q} \in \mathscr{R}(\mathbf{W}) \cap \mathscr{R}^{\perp}(\mathbf{Z})\},$$

i.e., the subspace $\mathscr{E}_0(\mathbf{I})$; cf. (3.6). ∎

Theorem 4.1 shows that, besides the algebraic representation (3.6), the subspace $\mathscr{E}_0(\mathbf{I})$ can be given a characterization, which is independent of the assumed dispersion matrix.

Combining Theorem 4.1 with Theorem 3.3 results in

COROLLARY 4.1. *The subspace* $\mathscr{E}_0(\mathbf{V}) \cap \mathscr{E}_0(\mathbf{I})$ *can be characterized as*

$$\mathscr{E}_0(\mathbf{V}) \cap \mathscr{E}_0(\mathbf{I}) = \{\mathbf{p}'\gamma \in \mathscr{E}_a : \mathbf{p}'\hat{\gamma} = \mathbf{p}'\hat{\gamma}_a \text{ and } \mathbf{p}'\tilde{\gamma} = \mathbf{p}'\tilde{\gamma}_a\}. \tag{4.5}$$

In order to study the dispersion-matrix robustness of the OLSEs, we introduce the subspaces

$$\tilde{\mathscr{E}} = \{\mathbf{p}'\gamma \in \mathscr{E} : \mathbf{p}'\tilde{\gamma} = \mathbf{p}'\hat{\gamma}\} \tag{4.6}$$

and

$$\tilde{\mathscr{E}}_a = \{\mathbf{p}'\gamma \in \mathscr{E}_a : \mathbf{p}'\tilde{\gamma}_a = \mathbf{p}'\hat{\gamma}_a\}. \tag{4.7}$$

For these we establish

LEMMA 4.1. *The subspaces* $\tilde{\mathscr{E}}$ *and* $\tilde{\mathscr{E}}_a$ *are given by*

$$\tilde{\mathscr{E}} = \{\mathbf{p}'\gamma : \mathbf{p} = \mathbf{W}'\mathbf{q},\ \mathbf{q} \in \mathscr{R}^{\perp}(\mathbf{V}\mathbf{W}^{\perp}) \cap \mathscr{R}(\mathbf{W})\}, \tag{4.8}$$

$$\tilde{\mathscr{E}}_a = \left\{\mathbf{p}'\gamma : \mathbf{p} = \mathbf{W}'\mathbf{q},\ \mathbf{q} \in \mathscr{R}^{\perp}\left[\mathbf{V}(\mathbf{W}:\mathbf{Z})^{\perp}\right] \cap \mathscr{R}(\mathbf{W}:\mathbf{Z}) \cap \mathscr{R}^{\perp}(\mathbf{Z})\right\}. \tag{4.9}$$

Proof. On account of (2.2) and (4.3), $\mathbf{p}'\gamma \in \tilde{\mathscr{E}}_a$ if and only if $\mathbf{p} = \mathbf{W}'\mathbf{q}$ for some $\mathbf{q} \in \mathscr{R}^{\perp}(\mathbf{Z})$ and

$$\operatorname{Cov}\left[\mathbf{q}'\mathbf{P}_{(\mathbf{W}:\mathbf{Z})}\mathbf{y}, (\mathbf{W}:\mathbf{Z})^{\perp\prime}\mathbf{y}\right] = \sigma^2\mathbf{q}'\mathbf{P}_{(\mathbf{W}:\mathbf{Z})}\mathbf{V}(\mathbf{W}:\mathbf{Z})^{\perp} = \mathbf{0}; \tag{4.10}$$

cf., e.g., Rao (1973a, pp. 317–318). Applying Lemma 2.1 and (2.20) to (4.10) yields the minimal representation (4.9) of $\tilde{\mathscr{E}}_a$. The representation (4.8) of $\tilde{\mathscr{E}}$ is obtained from (4.9) by choosing $\mathbf{Z} = \mathbf{0}$. ■

The following theorem provides an algebraic representation as well as characterizations of the subspace

$$\tilde{\mathscr{E}} \cap \mathscr{E}_a = \{\mathbf{p}'\gamma \in \mathscr{E}_a : \mathbf{p}'\tilde{\gamma} = \mathbf{p}'\hat{\gamma}\}, \tag{4.11}$$

along with algebraic representations of the subspaces

$$\tilde{\mathscr{E}}_a \cap \mathscr{E}_0(\mathbf{I}) = \{\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a : \mathbf{p}'\tilde{\boldsymbol{\gamma}} = \mathbf{p}'\hat{\boldsymbol{\gamma}}_a = \mathbf{p}'\tilde{\boldsymbol{\gamma}}_a\} \tag{4.12}$$

and

$$\tilde{\mathscr{E}}_a \cap \mathscr{E}_0(\mathbf{V}) = \{\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a : \mathbf{p}'\hat{\boldsymbol{\gamma}} = \mathbf{p}'\hat{\boldsymbol{\gamma}}_a = \mathbf{p}'\tilde{\boldsymbol{\gamma}}_a\}. \tag{4.13}$$

THEOREM 4.2. *For the subspaces $\tilde{\mathscr{E}} \cap \mathscr{E}_a$, $\tilde{\mathscr{E}}_a \cap \mathscr{E}_0(\mathbf{I})$, and $\tilde{\mathscr{E}}_a \cap \mathscr{E}_0(\mathbf{V})$ the following results hold*:

$$\tilde{\mathscr{E}} \cap \mathscr{E}_a = \{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p} = \mathbf{W}'\mathbf{q}, \mathbf{q} \in \mathscr{R}^{\perp}(\mathbf{V}\mathbf{W}^{\perp}) \cap \mathscr{R}(\mathbf{W}) \cap \mathscr{R}^{\perp}(\mathbf{Z})\} \tag{4.14}$$

$$= \mathscr{E}_0(\mathbf{V}) \cap \mathscr{E}_0(\mathbf{I}) \tag{4.15}$$

$$= \{\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a : \mathbf{p}'\tilde{\boldsymbol{\gamma}} = \mathbf{p}'\hat{\boldsymbol{\gamma}} = \mathbf{p}'\hat{\boldsymbol{\gamma}}_a = \mathbf{p}'\tilde{\boldsymbol{\gamma}}_a\}, \tag{4.16}$$

$$\tilde{\mathscr{E}}_a \cap \mathscr{E}_0(\mathbf{I}) = \left\{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p} = \mathbf{W}'\mathbf{q}, \mathbf{q} \in \mathscr{R}^{\perp}\left[\mathbf{V}(\mathbf{W}:\mathbf{Z})^{\perp}\right] \cap \mathscr{R}(\mathbf{W}) \cap \mathscr{R}^{\perp}(\mathbf{Z})\right\}, \tag{4.17}$$

$$\tilde{\mathscr{E}}_a \cap \mathscr{E}_0(\mathbf{V}) = \{\mathbf{p}'\boldsymbol{\gamma} : \mathbf{p} = \mathbf{W}'\mathbf{q}, \mathbf{q} \in \mathscr{R}^{\perp}(\mathbf{V}\mathbf{W}^{\perp}) \cap \mathscr{R}(\mathbf{W}:\mathbf{Z}) \cap \mathscr{R}^{\perp}(\mathbf{Z})\}. \tag{4.18}$$

Proof. The equality (4.14) follows from (4.8) and (2.2), and (4.15) is a consequence of (4.14), in view of (3.2), (3.6), and (1.1). The characterization (4.16) is obtained from (4.15) upon combining (4.11) with the characterization (4.5). The equality (4.17) follows from (4.9) and (3.6), whereas (4.18) is a consequence of (4.9), (3.2), and (1.1). ■

Comparing (4.14) with (4.9) it is seen that $\tilde{\mathscr{E}} \cap \mathscr{E}_a \subset \tilde{\mathscr{E}}_a$, in general, i.e.,

$$\{\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a : \mathbf{p}'\tilde{\boldsymbol{\gamma}} = \mathbf{p}'\hat{\boldsymbol{\gamma}}\} \subset \{\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a : \mathbf{p}'\tilde{\boldsymbol{\gamma}}_a = \mathbf{p}'\hat{\boldsymbol{\gamma}}_a\}. \tag{4.19}$$

Whereas (4.19) is related to dispersion-matrix robustness, the characterization (4.15) is crucial in combining dispersion-matrix robustness with robustness against nuisance parameters, since it may be written in the form

$$\{\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a : \mathbf{p}'\tilde{\boldsymbol{\gamma}} = \mathbf{p}'\hat{\boldsymbol{\gamma}}\} = \{\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a : \mathbf{p}'\hat{\boldsymbol{\gamma}} = \mathbf{p}'\hat{\boldsymbol{\gamma}}_a \text{ and } \mathbf{p}'\tilde{\boldsymbol{\gamma}} = \mathbf{p}'\tilde{\boldsymbol{\gamma}}_a\},$$

in view of Corollary 4.1. From the above equation, or directly from (4.16), it is seen that it suffices to require that $\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a$ be such that $\mathbf{p}'\tilde{\boldsymbol{\gamma}} = \mathbf{p}'\hat{\boldsymbol{\gamma}}$, in order to obtain equality between all four estimators.

From the point of view of applications, the most interesting subset of $\mathscr{E}_a$ is given by

$$\tilde{\mathscr{E}} \cap \mathscr{E}_a \cap \mathscr{E}_0(\mathbf{V}) = \{\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a : \mathbf{p}'\tilde{\boldsymbol{\gamma}} = \mathbf{p}'\hat{\boldsymbol{\gamma}} = \mathbf{p}'\hat{\boldsymbol{\gamma}}_a\}, \tag{4.20}$$

where the computationally simple OLSE $\mathbf{p}'\tilde{\boldsymbol{\gamma}}$ continues to be BLUE under both $\mathscr{M}(\mathbf{V})$ and $\mathscr{M}_a(\mathbf{V})$. From the preceding it follows, however, that the set (4.20) necessarily coincides with (4.16).

As $\tilde{\mathscr{E}} \subset \mathscr{E}$ and $\tilde{\mathscr{E}}_a \subset \mathscr{E}_a$, the representations given in Lemma 4.1 may be used to derive necessary and sufficient conditions for equality between $\tilde{\mathscr{E}}$ and $\mathscr{E}$ or between $\tilde{\mathscr{E}}_a$ and $\mathscr{E}_a$. The former case has been treated extensively in the literature as the "OLSE-versus-BLUE problem"; see Rao (1967, 1968) and Zyskind (1967) for conditions under the most general setup, and also Puntanen and Styan (1989) for a comprehensive survey. For the latter case we establish

THEOREM 4.3. *The subspaces $\tilde{\mathscr{E}}_a$ and $\mathscr{E}_a$ coincide if and only if*

$$\mathscr{R}\left[\mathbf{V}(\mathbf{W}:\mathbf{Z})^{\perp}\right] \subset \mathscr{R}^{\perp}(\mathbf{W}:\mathbf{Z}) + \mathscr{R}(\mathbf{Z}). \tag{4.21}$$

Proof. From (2.2) it follows that

$$\dim \mathscr{E}_a = \dim \mathscr{R}(\mathbf{W}'\mathbf{Q}_{\mathbf{Z}}) = \dim \mathscr{R}(\mathbf{Q}_{\mathbf{Z}}\mathbf{W}),$$

so that

$$\dim \mathscr{E}_a = \dim \mathscr{R}(\mathbf{W}:\mathbf{Z}) \cap \mathscr{R}^{\perp}(\mathbf{Z}), \tag{4.22}$$

in view of Lemma 2.1. Comparing (4.22) with (4.9) it is seen that $\tilde{\mathscr{E}}_a = \mathscr{E}_a$ if and only if

$$\mathscr{R}(\mathbf{W}:\mathbf{Z}) \cap \mathscr{R}^{\perp}(\mathbf{Z}) \subset \mathscr{R}^{\perp}\left[\mathbf{V}(\mathbf{W}:\mathbf{Z})^{\perp}\right],$$

or equivalently, (4.21) holds. ■

Consider now the invariance condition

$$\mathscr{R}\left[\mathbf{V}(\mathbf{W}:\mathbf{Z})^{\perp}\right] \subset \mathscr{R}^{\perp}(\mathbf{W}:\mathbf{Z}), \tag{4.23}$$

which is well known as being necessary and sufficient for the OLSE of $E_a(\mathbf{y}) = \mathbf{W}\boldsymbol{\gamma} + \mathbf{Z}\boldsymbol{\delta}$ (or equivalently, for the OLSE of every functional of $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$) to be BLUE under $\mathscr{M}_a(\mathbf{V})$; cf. Rao (1967, p. 364) and Rao (1968, Lemma 1). A comparison of (4.23) with (4.21) shows precisely the extent to which the invariance condition (4.23) may be relaxed when estimators of functionals $\mathbf{p}'\boldsymbol{\gamma} \in \mathscr{E}_a$ involving only the main parameters $\boldsymbol{\gamma}$ are considered.

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